6-regular Frobenius circulants and Eisenstein-Jacobi networks

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  - An interconnection network is modelled by a graph
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  - etc.

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- We will consider two measures:
  - minimum gossiping time under the store-and-forward, all-port and full-duplex model
  - minimum edge-congestion for all-to-all routing
- What are the 'most efficient' graphs (of 'small' degree) with respect to these measures?

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- In general,

$$\pi_m(\Gamma) \neq \pi(\Gamma), \overrightarrow{\pi}_m(\Gamma) \neq \overrightarrow{\pi}(\Gamma)$$

$$\pi(\Gamma) \neq 2\overrightarrow{\pi}(\Gamma), \pi_m(\Gamma) \neq 2\overrightarrow{\pi}_m(\Gamma)$$

Trivial lower bounds

$$\pi_m(\Gamma) \geqslant \pi(\Gamma) \geqslant \frac{\sum_{u,v \in V} d(u,v)}{|E|}$$

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Question

A: *Which graphs can achieve these bounds?*

Gossiping

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Assume the store-and-forward, all-port and full-duplex model:

- a vertex must receive a message wholly before transmitting it to other vertices;
- 'all-neighbour transmission' at the same time step;
- bidirectional transmission on each edge;
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Definition

$$t(\Gamma) = \text{minimum time steps required}$$

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B: *Which graphs can achieve this bound?*

Cayley graphs

Definition

Let G be a group, and let $S \subset G$ be such that $1 \notin S$ and $s^{-1} \in S$ for all $s \in S$.

The Cayley graph $\mathrm{Cay}(G, S)$ is defined to have vertex set G such that $x, y \in G$ adjacent if and only if $xy^{-1} \in S$.

A circulant graph is a Cayley graph on a cyclic group.

Semidirect product

Definition

Let H and K be groups such that H acts on K (as a group) via a homomorphism $H \to \mathrm{Aut}(K)$.

The semidirect product of K by H, denoted $K.H$, is the group on $K \times H$ under the operation:

$$(k_1, h_1)(k_2, h_2) := (k_1 k_2^{h_1^{-1}}, h_1 h_2).$$

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Equivalently, if G is a group and

$$K \trianglelefteq G, H \leqslant G, G = HK, H \cap K = 1,$$

then $G = K.H$.

Frobenius groups

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Theorem

(Thompson 1959) A finite Frobenius group G on V has a nilpotent normal subgroup K (Frobenius kernel) which is regular on V. Thus

$$G = K.H \text{ (semidirect product)},$$

where H is the stabiliser of a point of V.

Frobenius graphs

Definition

(Solé 94, Fang-Li-Praeger 98)
Let $G = K.H$ be a finite Frobenius group. Let

$$S = \begin{cases} a^H, & |H| \text{ even or } |a| = 2 \quad \text{[first-kind]} \\ a^H \cup (a^{-1})^H, & |H| \text{ odd and } |a| \neq 2 \quad \text{[second-kind]} \end{cases}$$

for some $a \in K$ satisfying $\langle a^H \rangle = K$, where a^H is the H-orbit on K containing a.

Call $\mathrm{Cay}(K, S)$ a G-Frobenius graph.

Partial answer to Question A

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- d: diameter of $\mathrm{Cay}(K, S)$
- n_i: number of H-orbits of vertices at distance i from 1 in $\mathrm{Cay}(K, S)$, $i = 1, \ldots, d$

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Theorem

(Solé, Fang-Li-Praeger 98)
Let $\Gamma = \mathrm{Cay}(K, S)$ *be a Frobenius graph. Then*

$$\pi(\Gamma) = \frac{\sum_{u,v \in V} d(u, v)}{|E|} = \begin{cases} 2\sum_{i=1}^{d} in_i, & \textit{[first-kind]} \\ \sum_{i=1}^{d} in_i, & \textit{[second-kind]} \end{cases}$$

Theorem

(Zhou 09)
Let $\Gamma = \mathrm{Cay}(K, S)$ *be a first-kind* G*-Frobenius graph. Then there exists a routing which is*

(a) *a shortest path routing;*

(b) G*-arc transitive;*

(c) *both edge- and arc-uniform;*

(d) *optimal for* π, $\overrightarrow{\pi}$, $\overrightarrow{\pi}_m$, π_m *simultaneously.*

Moreover, if the H*-orbits on* K *are known, we can construct such routings in polynomial time. Furthermore, we have*

$$\pi(\Gamma) = 2\overrightarrow{\pi}(\Gamma) = 2\overrightarrow{\pi}_m(\Gamma) = \pi_m(\Gamma) = 2\sum_{i=1}^{d} in_i.$$

- Our algorithm can produce many routings satisfying (a)-(d).

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- The formula for $\overrightarrow{\pi}_m$ and a result of Diaconis-Stroock imply the following:

Corollary

Let Γ, d, n_i be as above. Then the spectral gap of Γ satisfies

$$|H| - \lambda_2(\Gamma) \geqslant \frac{|K|}{d \sum_{i=1}^{d} i n_i},$$

where λ_2 is the second largest eigenvalue.

Partial answer to Question B

Theorem

(Zhou 09)
Let $\Gamma = \mathrm{Cay}(K, S)$ *be a first-kind* G*-Frobenius graph. Then*

$$t(\Gamma) = \frac{|K| - 1}{|S|}.$$

Moreover, there exist optimal gossiping schemes such that

(a) *messages are always transmitted along shortest paths;*

(b) *at any time every arc is used exactly once for message transmission;*

(c) *at any time* $\geqslant 2$ *and for any vertex* g*, the set* $A(g)$ *of arcs transmitting the message originated from* g *is a matching of* Γ*, and* $\{A(g) : g \in K\}$ *is a partition of the arcs of* Γ*.*

Furthermore, if we know the H*-orbits on* K*, then we can construct such schemes in polynomial time.*

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- It would be good to construct concrete families of first-kind Frobenius graphs of small degree.
- A classification of 4-regular first-kind Frobenius circulants was given by Thomson and Zhou in 2008.
- We will focus on 6-regular first-kind Frobenius circulants.

6-regular circulants

- Let $n \geqslant 7$ and a, b, c be integers such that $1 \leqslant a, b, c \leqslant n-1$ and $a, b, c, n-a, n-b, n-c$ are pairwise distinct.
- The 'triple-loop' network $TL_n(a, b, c)$ is defined to have vertex set $\mathbb{Z}_n$ such that

$$x \sim x \pm a, x \sim x \pm b, x \sim x \pm c \pmod{n}.$$

- We consider $TL_n(a, b, 1)$ only.

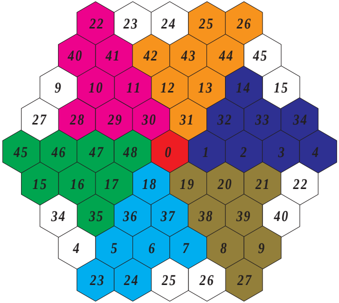
Geometric triple-loop network

Definition

(Yebra, Fiol, Morillo and Alegre 85) $TL_n(a, b, c)$ is geometric if

$$a' + b' + c' \equiv 0 \mod n$$

for some $a' \in \{a, n-a\}, b' \in \{b, n-b\}, c' \in \{c, n-c\}$.



Hexagonal tessellation of $TL_{49}(31, 1, 30)$

Rotational Cayley graphs

Definition

(Bermond, Kodate and Pérennes 96)
A complete rotation of $\mathrm{Cay}(K, S)$ is an automorphism of K which fixes S setwise and induces a cyclic permutation on S.

$\mathrm{Cay}(K, S)$ is rotational if it admits a complete rotation.

Theorem

(Thomson and Zhou 08-12+)

Let $n \geqslant 7$ be an integer.

Then there exists a 6-regular first kind Frobenius circulant $TL_n(a, b, 1)$ of order n (with cyclic kernel) if and only if $n \equiv 1 \mod 6$ and

$$x^2 - x + 1 \equiv 0 \mod n \tag{1}$$

has a solution.

Theorem

(cont'd)

Moreover, if this condition holds, then

(a) *each prime factor of n is congruent to 1 modulo 6;*

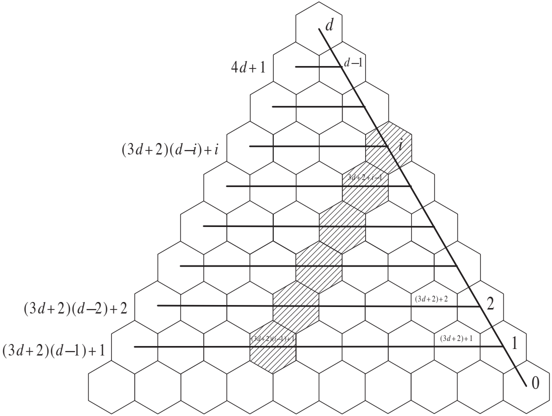
(b) *each solution a to the equation above gives rise to a 6-regular first kind Frobenius circulant $TL_n(a, b, 1)$, and vice versa, with $b \equiv a - 1 \mod n$; $TL_n(a, a-1, 1)$ is a rotational, geometric, $\mathbb{Z}_n.H$-arc-transitive and first-kind $\mathbb{Z}_n.H$-Frobenius graph admitting $[a]$ and $-[a^2]$ as complete rotations, where*

$$H = \langle [a] \rangle = \{\pm[1], \pm[a], \pm[a-1]\}; \qquad (2)$$

(c) *there are exactly 2^{l-1} pairwise non-isomorphic 6-regular first kind Frobenius circulants of order n, where l is the number of distinct prime factors of n.*

Optimal routing, gossiping and broadcasting

- We gave optimal routing and gossiping schemes for $TL_n(a, a-1, 1)$ by specifying our general theory and using knowledge of H-orbits on $\mathbb{Z}_n$.
- Such knowledge was obtained through a link with Eisenstein-Jacobi networks.
- Formula for edge-forwarding index is messy.
- Gossiping time $= (n-1)/6$
- Broadcasting time $=$ diameter $+$ (2 or 3)



An illustration of optimal routing and gossiping: Constructing the ‘canonical’ spanning tree T_0 for $TL_{n_d}(3d+1, 1, -(3d+2))$, where $n_d = 3d^2 + 3d + 1$.

HARTS (hexagonal meshes)

- A distributed real-time computing system [Chen, Shin and Kandlur, *IEEE Trans. Computers* 39 (1) (1990) 10-18].
- Physically built at the Real-Time Computing Laboratory, The University of Michigan.
- Properties studied in [Dolter, Ramanathan and Shin, *IEEE Trans. Computers*, 40 (6) (1991) 669-680] and [Albader, Bose and Flahive, *IEEE Trans. Parallel Distrib. Syst.* 23 (1) (2012) 69-77]
- It belongs to the family of 6-regular first-kind Frobenius circulants.
- Actually it is not new.

- Denote $n_k = 3k^2 + 3k + 1$, where $k \geqslant 2$.
- (Yebra, Fiol, Morillo and Alegre 85) $TL_{n_k} = TL_{n_k}(3k+2, 3k+1, 1)$ has the maximum possible order among all 6-regular geometric circulants of diameter k.
- (Thomson and Zhou 10) TL_{n_k} is a first-kind Frobenius graph.
- The HARTS H_k of size k has diameter $k-1$ and n_{k-1} vertices [CSK], and is isomorphic [CSK, ABF] to the circulant $\mathrm{Cay}(\mathbb{Z}_{n_{k-1}}, S)$, where

$$S = \{\pm[k-1], \pm[k], \pm[2k-1]\}.$$

- $H_k \cong TL_{n_{k-1}}$

EJ networks

Another research group who came up with a closed related family of graphs:

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- $\rho = (1 + \sqrt{-3})/2$
- $\mathbb{Z}[\rho] = \{x + y\rho : x, y \in \mathbb{Z}\}$ (Eisenstein-Jacobi integers)
- $\alpha = a + b\rho \in \mathbb{Z}[\rho] \setminus \{0\}$
- $N(\alpha) = a^2 + ab + b^2$ (norm)
- $\mathbb{Z}[\rho]_\alpha = \mathbb{Z}[\rho]/(\alpha)$
- $H_\alpha = \{\pm[1]_\alpha, \pm[\rho]_\alpha, \pm[\rho^2]_\alpha\}$

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Definition

EJ network: $EJ_\alpha = \mathrm{Cay}(\mathbb{Z}[\rho]_\alpha, H_\alpha)$

Theorem

(Thomson and Zhou 08-12+)

$$\begin{aligned} & \{\textit{6-regular first kind Frobenius circulants}\} \\ = \; & \{EJ_{a+b\rho} : N(a+b\rho) \equiv 1 \mod 6, \textit{ a and b coprime}\} \end{aligned}$$

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$$\begin{aligned} & \{\textit{6-regular first kind Frobenius circulants}\} \\ = \; & \{EJ_{a+b\rho} : N(a+b\rho) \equiv 1 \mod 6, \textit{ a and b coprime}\} \end{aligned}$$

We do not know whether non-Frobenius EJ networks also admit 'perfect' routing, gossiping and broadcasting schemes.

Covers

Theorem

(Thomson and Zhou 08-12+)
Let $\alpha, \beta \in \mathbb{Z}[\rho]$ be nonzero such that $N(\alpha) \geqslant 7$.

Then $EJ_{\alpha\beta}$ is an $N(\beta)$-fold cover of EJ_α and can be constructed from EJ_α.

Corollary

(Thomson and Zhou 08-12+)
Let $\alpha = c + d\rho \in \mathbb{Z}[\rho]$ with $7 \leqslant N(\alpha) \equiv 1 \mod 6$ that is not an associate of any real integer. Denote

$$\ell = \mathtt{gcd}(c, d),\ c' = c/\ell,\ d' = d/\ell,\ \alpha' = c' + d'\rho.$$

Then EJ_α is an ℓ^2-fold cover of a 6-regular first-kind Frobenius circulant that is isomorphic to $EJ_{\alpha'}$.

Theorem

(Thomson and Zhou 08-12+)
Let $n \geqslant 7$ be an integer all of whose prime factors are congruent to 1 modulo 6.

Let m be a proper divisor of n.

Then any first-kind Frobenius $TL_n(a, a-1, 1)$ is an n/m-fold cover of a smaller first-kind Frobenius $TL_m(a_m, a_m - 1, 1)$.

Rotational Cayley graphs

A complete rotation of $\mathrm{Cay}(K, S)$ is an automorphism ω of K which fixes S setwise and induces a cyclic permutation on S.

An element $g \in K$ is a fixed point of ω if $g \neq 1$ and there exists i such that $g^{\omega^i} = g$.

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Theorem

(Bermond, Kodate and Pérennes 1996)
If $\mathrm{Cay}(K, S)$ *admits a complete rotation whose fixed point set is an independent set and not a vertex-cut, then*

$$t(\mathrm{Cay}(K, S)) = \left\lceil \frac{|K| - 1}{|S|} \right\rceil.$$

Theorem

(Zhou 2009)
Let $\mathrm{Cay}(K, S)$ *be a connected Cayley graph. Suppose that there exists* $H \leqslant \mathrm{Aut}(K)$ *that*

- H *fixes* S *setwise and is regular on* S*;*
- $K \backslash (\{x \in K : H_x = 1\} \cup \{1\})$ *is an independent set and not a vertex-cut of* Γ*.*

Then

$$t(\mathrm{Cay}(K, S)) = \left\lceil \frac{|K| - 1}{|S|} \right\rceil.$$

Classification of rotational first-kind Frobenius circulants

Theorem

(A. Thomson and S. Zhou 2013+)
Let $n = p_1^{e_1} \dots p_l^{e_l}$ *and* $D = \mathtt{gcd}(p_1 - 1, \dots, p_l - 1)$.

(a) $\exists$ *a rotational first-kind Frobenius circulant with kernel* $\mathbb{Z}_n$ *and degree* d *iff* n *is odd and* d *is an even divisor of* D.

(b) $\varphi(d)^{l-1}$ *such circulants (pairwise non-isomorphic)*

(c) *Each is isomorphic to* $\mathrm{Cay}(\mathbb{Z}_n, \langle [h] \rangle)$, *where* $h = \sum_{i=1}^{l} \frac{n}{p_i^{e_i}} b_i h_i$, *with* $b_i(n/p_i^{e_i}) \equiv 1 \pmod{p_i^{e_i}}$ *and* $h_i \equiv \eta_i^{m_i \varphi(p_i^{e_i})/d} \pmod{p_i^{e_i}}$ *for a fixed primitive root* η_i *modulo* $p_i^{e_i}$ *and an integer* m_i *coprime to* d.

Balanced regular Cayley maps

Definition

A map is a 2-cell embedding of a connected graph on an orientable surface.

A cyclic permutation ρ of S induces a natural embedding of $\mathrm{Cay}(G, S)$, giving a Cayley map $M = CM(G, S, \rho)$.

M is balanced if $\rho(s^{-1}) = \rho(s)^{-1}$ for $s \in S$, and regular if $\mathrm{Aut}(M)$ is regular on the set of arcs of $\mathrm{Cay}(G, S)$.

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Complete rotation in a Cayley graph $\leftrightarrow$ 2-cell embedding on a closed orientable surface as a balanced regular Cayley map

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A map is a 2-cell embedding of a connected graph on an orientable surface.

A cyclic permutation ρ of S induces a natural embedding of $\mathrm{Cay}(G, S)$, giving a Cayley map $M = CM(G, S, \rho)$.

M is balanced if $\rho(s^{-1}) = \rho(s)^{-1}$ for $s \in S$, and regular if $\mathrm{Aut}(M)$ is regular on the set of arcs of $\mathrm{Cay}(G, S)$.

Complete rotation in a Cayley graph $\leftrightarrow$ 2-cell embedding on a closed orientable surface as a balanced regular Cayley map

Theorem

(A. Thomson and S. Zhou 2013+)
We know exactly when a first-kind Frobenius circulant can be embeded on a closed orientable surface as a balanced regular Cayley map.

Mendelsohn through Frobenius

Definition

(a) A (v, k, λ)-Mendelsohn design, (v, k, λ)-MD, consists of a set X of v points and a collection $\mathcal{B}$ of cyclically ordered k-subsets (blocks) of X such that every ordered pair of points are consecutive in exactly λ blocks.

(b) A (v, k, λ)-MD $(X, \mathcal{B})$ is ℓ-fold perfect if, for $t = 1, \ldots, \ell$, every ordered pair of elements of X appears t-apart in exactly λ blocks. A (v, k, λ)-MD is perfect, (v, k, λ)-PMD, if it is $(k-1)$-fold perfect.

(c) A (v, k, λ)-MD is resolvable, (v, k, λ)-RMD, if $v \equiv 0 \mod k$ and the set of blocks can be partitioned into $\lambda(v-1)$ parts each containing v/k pairwise disjoint blocks, or $v \equiv 1 \mod k$ and the set of blocks can be partitioned into λv parts each containing $(v-1)/k$ pairwise disjoint blocks.

Theorem

(F. D. Hsu and S. Zhou 2013+)

(a) *A $(v, k, 1)$-RMD exists for all integers $v \geqslant 3, k \geqslant 2$ with $v \equiv 1 \mod k$ such that there exist a finite Frobenius group $K.H$ with order $|K| = v$ and an element ϕ of H with order k.*

(b) *This $(v, k, 1)$-RMD is $(p(k) - 1)$-fold perfect, where $p(k)$ is the smallest prime factor of k.*

(c) *If k is a prime, then it is a $(v, k, 1)$-RPMD.*

Theorem

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(c) *If k is a prime, then it is a $(v, k, 1)$-RPMD.*

Theorem

(F. D. Hsu and S. Zhou 2013+)

Let $v = p_1^{e_1} \ldots p_t^{e_t} \geqslant 3$. A $(v, k, 1)$-RPMD exists for every prime factor k of $\texttt{gcd}(p_1^{e_1} - 1, \ldots, p_t^{e_t} - 1)$.

Corollary

(F. D. Hsu and S. Zhou 2013+)
Let k be a fixed prime.
For any primes $p_1, \ldots, p_t \equiv 1 \mod k$ and any integers $e_1, \ldots, e_t \geqslant 1$, there exists a $(p_1^{e_1} \ldots p_t^{e_t}, k, 1)$-RPMD.

By the well known Dirichlet prime number theorem, there are infinitely many primes congruent to 1 modulo k.

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THANK YOU